International Journal of Information Technology & Decision Making
Vol. 15, No. 4 (2016) 839–859
© World Scientific Publishing Company
DOI: 10.1142/S0219622016400058

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Voting with Intensity of Preferences

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Published 25 July 2016

In this paper, we develop a method based on the idea of pairwise voting to rank projects or candidates and incorporate in the ranking process how strongly the referees/voters feel about the comparisons they make. Voting is a modified form of ranking and all the votes are equally important. However, there are situations similar to voting in which the votes are not just ordinal but each voter expresses an intensity of preference for the different candidates, e.g., ranking projects for funding. We show that our method yields the same results as ordinal voting in large populations when the intensity of preferences becomes extreme. Voting with intensity of preferences does not violate democracy but soften the stand of voters and allows for consideration of the diversity of issues involved in voting.

Keywords: Voting; group decision making; pairwise comparisons; reciprocal matrices; analytic hierarchy process.

1. Introduction

One of the purposes of the electoral process is to elicit and measure the preference of voters. However, the use of voters' preferences is not limited to the election of government officials or board members in the public and private sector. The voting process is also used to rank projects, e.g., in the National Science Foundation or the National Institutes of Health (NIH), and in general, to rank alternative courses of action. In many cases, ranking from the voters' preferences needs to be translated into priorities that can be used to allocate resources, such as prioritizing spending in a portfolio of research projects, when not all the projects can be funded due to budgetary constraints. In general, NIH does not have all the necessary funds to support all the projects requesting funding. Referees are then required to rank the projects making intensity of preference statements according to some pre-established dimensions, such as innovation, the institution, the researcher past contributions, and so on, that will yield a cardinal ranking of the projects. In the end, the projects are assigned a score that help the referees decide how to fund them.

However, the process of ranking projects to allocate resources to them is not really a voting problem unless all the voters are considered, not as an average but

as a totality. Thus, when considering the possibility of voting with intensity of preferences, one could be thinking that some voters may have more power than others as it happens with voting for an issue in a shareholders' meeting where the shareholder with more shares has more power than one with just a few shares.

In this paper, we develop a method based on the idea of pairwise voting to rank projects or candidates and incorporate in the ranking process how strongly the voters feel about the comparisons they make, but assuming that all the voters have the same power. We show below that voting with intensity of preferences using the eigenvector method of pairwise comparisons does not violate democracy. The rest of the paper is structured as follows. In Sec. 2, we summarize the literature of pairwise voting methods and introduce the notation used in the paper. In Sec. 3, we introduce the eigenvector pairwise voting method and its mathematical properties. In Sec. 4, we introduce intensity of preference in voting under the assumption that all voters have equal power, and in Sec. 5 we show conclusions.

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2. Background

The application of measuring voting preferences has been used in some form or another for centuries dating back at least to the 13th century.[1] Some of the earliest recorded methods are still prevalent today, and they are even considered the main divide between the approaches,[2] they were developed in France in the 1700's by early political scientists and mathematicians Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet,[3] and Jean-Charles, Chevalier de Borda.[4]

Condorcet developed a framework to evaluate election results that would help ensure that the "*most likely to be correct*" candidate was elected. As pairwise votes are tallied, there is a positive probability for cyclical majorities to occur; in the case of three candidates a cyclical outcome would be $a_1 \succ a_2, a_2 \succ a_3$ and $a_3 \succ a_1$ where " $\succ$ " means "more preferred than". As the number of possible orderings increases, the probability of similar cycles occurring increases. To eliminate these preference cycles, Condorcet proposed a simple solution termed "Condorcet Criterion" that requires the "true" winner to have a majority over every other candidate.[2,5] A downfall of this criterion is that it may eliminate certain preference orderings that could have a greater number of overall votes.

Borda was also concerned about voting methods. He proposed a method of assigning increasing scores beginning with zero to the least preferred candidate and higher scores to more preferred candidates. The scores are tallied for each individual candidate and the candidate with the highest score wins the election. One disadvantage of Borda's method is that a simple majority may not reflect the overall "true" preference; however, a similar disadvantage can occur when a mediocre but broadly accepted candidate receives the greatest number of points. According to Saari,[6] Borda's counting method is superior to Condorcet rankings; so whenever a discrepancy appears, between the Borda and Condorcet method, the evidence supports Borda's method.

We use the following sample ballot matrix or voting profile B to illustrate the results and differences between the voting methods:

$$B = \begin{array}{c} \begin{array}{cccc} (4) & (3) & (5) & (2) \end{array} \\ \begin{pmatrix} a_1 & a_2 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_1 & a_2 & a_1 \end{pmatrix} \end{array}.$$

The columns in B represent the order in which the voters prefer the alternatives and the number between parentheses above the orderings is the number of voters that prefer that ordering, e.g., four voters have the preference ordering $a_1 \succ a_2 \succ a_3$.

Borda's winner in the profile B is candidate 1. The outcomes from B suggest that candidate 1 should be Condorcet's winner because candidate 1 is ranked ahead of candidates 2 and 3 and candidates 2 and 3 are ranked ahead of each other the same number of times. Borda's method selects the candidate with overall greater weight, but Condorcet's method avoids mediocrity. So the question is: How can one derive advantages from both methods? Scholars have proposed interesting solutions that build on the basic criteria from those early methods and try to solve other problems or paradoxes.[7]

Another key distinction within the different counting methods is how many candidates or options the voter has. Some methods only allow the voter to select for a single option; the plurality and run-off methods are two examples. Other methods either allow the voter to select or rank multiple candidates. In this paper, we focus on the latter methods that are broadly referred to as preferential voting.

In the next section, we introduce the notation and basic terminology, and briefly review four of the more relevant pairwise voting methods, namely, Kemeny–Young, Ranked Pairs, Copeland and Schulze. In Sec. 4, we introduce the eigenvector method of pairwise voting (EMPV) based on the solution of an eigenvalue problem. In Sec. 5 we extend the method to consider cardinal intensity of preferences also based on pairwise ratios.

2.1. *Notation and terminology*

Let $\mathsf{A} = \{a_1, \ldots, a_m\}$ be a set of alternatives. Let $N = \{1, 2, 3, \ldots\}$ be the set of voters. A *preference order* $\sigma = \{a_{i_1}, \ldots, a_{i_m}\}$ is a ranking of the alternatives. Let $L(\mathsf{A})$ be the set of all m! orders. A *profile* on a group of voters $M \subset_{\neq} N$ is a mapping $\phi : M \to L(\mathsf{A})$. Let Φ be the set of all the possible profiles. For $\sigma \in L(\mathsf{A})$ and $\phi \in \Phi$, let $n_\sigma(\phi)$ be the number of voters in the profile ϕ that have the preference order σ. A *preference function* is a mapping from the set of profiles Φ to the set of preference orders, $f : \Phi \to L(\mathsf{A})$. A *choice function* is a mapping from the set of profiles Φ to the set of nonempty subsets of A. Let $(\sigma_1(\phi), \ldots, \sigma_H(\phi))$ be the different orderings in the profile ϕ. Let $v_{ij}[\sigma_h(\phi)]$ be the number of voters who prefer i to j in the ordering $\sigma_h(\phi)$ of profile ϕ. Let $n_{\sigma_h}(\phi)$ be number of votes in the ordering $\sigma_h(\phi)$ of profile ϕ.

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Let $w_i[\sigma_h(\phi)]$ be the weight assigned to the ith alternative in the ordering $\sigma_h(\phi)$ of profile ϕ.

2.2. *Borda's method*

Let $\sigma_h(\phi) = (i_{[h:1]}, \ldots, i_{[h:n]})$where $i_{[h:s]} > i_{[h:s+1]}$ for all $s = 1, \ldots, n-1$. Borda's method assigns weights as follows: zero to the least preferred alternative, i.e., $w_{[n:n]}[\sigma_h(\phi)] = 0$, one to the next-to-least preferred alternative, two to the next least preferred alternative, and so on, i.e., for the ith least preferred alternative, $w_{[i:n]}[\sigma_h(\phi)] = n - i$. Borda's method yields the alternative given by

$$a_B = \arg\left\{\max_i \sum_{h=1}^{H} w_{[i:n]}[\sigma_h(\phi)] * n_{\sigma_h}(\phi)\right\}.$$

For the profile B given above, Borda's alternative a_B is alternative to a_1 as shown in Table 1.

2.3. *Condorcet's method*

Condorcet's method yields the alternative a_C that defeats all other alternatives pairwise, i.e., $v_{Cj}[\sigma_h(\phi)] > v_{jC}[\sigma_h(\phi)]$ for all j and h. Let

$$n_{ij}(\phi) = \sum_{h=1}^{H} \{v_{ij}[\sigma_h(\phi)] - v_{ji}[\sigma_h(\phi)]\}.$$

Let $I_{n_{ij}(\phi)>0}(i)$ be the indicator variable for $\sum_{h=1}^{H} v_{ij}[\sigma_h(\phi)] > \sum_{h=1}^{H} v_{ji}[\sigma_h(\phi)]$. Condorcet's method yields the alternative a_C that satisfies $\sum_{\substack{j=1 \\ j \neq C}}^{n} I_{n_{Cj}(\phi)>0}(C) = n - 1$.

For the profile B given above, Condorcet's alternative a_C also coincides with alternative a_1 as can be seen in Table 2.

Table 1. Borda's method.

	(4)	(3)	(5)	(2)	Total
a_1	8	0	10	1	19
a_2	4	6	0	0	10
a_3	0	3	5	2	10

Table 2. Condorcet's method.

$n_{ij}(B) = v_{ij}(B) - v_{ji}(B)$	a_1 a_2 a_3	$\sum_{\substack{j=1 \\ j \neq 1}}^{3} I_{n_{ij}(\phi)>0}(i)$
a_1	$\begin{pmatrix} 0 & 11-3 & 9-5 \\ 3-11 & 0 & 7-7 \\ 5-9 & 7-7 & 0 \end{pmatrix}$	$2 \leftarrow a_C$
a_2		0
a_3		0

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Definition 1. Definition A preference function $f : \Phi \to L(\mathsf{A})$ satisfies the *Condorcet criterion* if and only if for any profile $\phi \in \Phi$ and any two alternatives i and j, $n_{ij}(\phi) > 0$, i.e., the preference order in which j is preferred to i does not belong to $f(\phi)$.

2.4. *Kemeny–Young's method*

Kemeny[8] introduced a preference function and measurements of distance between preferences. For a given preference order $\sigma \in L(\mathsf{A})$, let $X^\sigma = \{x_{ij}^\sigma\}$ be the election matrix for a preference order σ where

$$x_{ij}^\sigma = \begin{cases} 1 & \text{if } a_i \text{ f } a_j \\ 0 & \text{if } a_i : a_j \\ -1 & \text{if } a_j \text{ f } a_i \end{cases}$$

The election matrix for a profile is given by $X^\phi = \sum_{\forall \sigma \in \phi} n_\sigma(\phi) X^\sigma$. Kemeny's method calculates either the median or the mean ranking to find the ordering that is closest, in the sense of a metric, from the individuals' orderings. Young and Levenglick recommend using the median ranking because it leads to a Condorcet method.[2] The median ranking is given by the solution of the optimization problem $\min_\tau \sum_{\sigma \in \phi} n_\sigma(\phi) \frac{1}{2} \sum_{i,j} \left| x_{ij}^\sigma - x_{ij}^\tau \right|$, while the mean ranking is given by the solution of the optimization problem

$$\min_\tau \sum_{\sigma \in \phi} n_\sigma(\phi) \left(\frac{1}{2} \sum_{i,j} \left| x_{ij}^\sigma - x_{ij}^\tau \right| \right)^2.$$

For the profile given by B, both the median and the mean ranking solutions yield $a_1 \succ a_2 \succ a_3$.

2.5. *Ranked Pairs method*

"Clones" is the term Tideman uses to describe alternatives that are "arbitrarily close to one another in some shared perceptual space."[9] Tideman developed the Ranked-Pairs Rule to "almost always" eliminate the concern of clones while still satisfying the Condorcet criterion. Candidates are ranked according to the voter's preferences (ties are allowed). For a given profile ϕ, a pairwise matrix Y^ϕ is defined in terms of totals representing how many times a specific candidate was preferred over another, i.e., $x_{ij}^\phi = v_{ij}(\phi)$. Let $Z = (z_{ij})$ be an incidence matrix defined from the matrix X^ϕ for a directed graph as follows:

$$z_{ij} = \begin{cases} 1 & \text{if } v_{ij}(\phi) > v_{ji}(\phi) \\ 0 & \text{otherwise.} \end{cases}$$

Powers of Z yield the number of paths between pairs of alternatives. If $z_{ii} \neq 0$ for some i then there are ambiguities (or violation of transitivity), i.e., there are loops. Loops are eliminated by deleting the arc corresponding to the pairwise comparison with the smallest number of votes in the loop. Once the graph has no loops, *the final*

ranking is given by the outflow index of the nodes (alternatives). For the profile B, we have

$$X^{\phi} = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 0 & 11 & 9 \\ 3 & 0 & 7 \\ 5 & 7 & 0 \end{pmatrix} \end{array} Z = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0(1) \\ 0 & 1(0) & 0 \end{pmatrix} \end{array}$$

Note that Z is not unique because there is a tie between alternatives 2 and 3. Since there are no loops, the final ranking is given by $a_1 > a_2 \sim a_3$.

2.6. *Copeland's method*

This method selects the alternative with the most pairwise victories, i.e., it yields the alternative that satisfies

$$\max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} I_{n_{ij}(\phi)>0}(i).$$

This method is similar to Condorcet's method except that it does not require an overall victory against every other candidate, i.e.,

$$\max_i \sum_{\substack{j=1 \\ j \neq i}}^{n} I_{n_{ij}(\phi)>0}(i)$$

need not equal $n-1$. Different point systems have evolved for assigning points to the pairwise comparison winner; perhaps the simplest method is to allot 1 point to the winner and no points are allotted to the loser or to either alternatives when a tie occurs. The points are added up and the winner declared.

2.7. *Schulze's method*

Schulze's method, developed by Schulze,[10] is a single winner preferential voting method that complies with Condorcet's criterion. Schulze assumes that each voter will cast at least a partial ballot. Schulze defines a "path" from alternative a_i to a_j as an ordered set of alternatives $a_{i_1} \cdots a_{i_k}$ where $a_{i_1} \equiv a_i$ and $a_{i_k} \equiv a_j$, and $v_{i_h,i_{h+1}}(\phi) > v_{i_{h+1},i_h}(\phi)$, $h = 1, \ldots, k-1$. The strength of a path of length k from alternative a_i to a_j is given by $\min_{1 \le h \le k}\{v_{i_h,i_{h+1}}(\phi)\}$. The strongest path from alternative a_i to a_j is given by

$$p[a_i, a_j] = \max_{\text{all paths from i to j}} \left\{ \min_{1 \le h \le k} \{v_{i_h,i_{h+1}}(\phi)\} \right\}.$$

A candidate a_i is considered a "potential winner" *if and only if* $p[a_i, a_j] \ge p[a_j, a_i]$ for all j. To deal with ties Schulze explains that a single potential winner is the

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unique winner. Otherwise, a "Tie Breaking Ranking of Candidates" (TBRC) is performed by choosing a random ballot, and using the respective profile to rank the potential winners, and thus, identifying the winner. For the profile B in the example, we have $a_1 : a_3 \text{ f } a_2$, and similar to the other methods a_1 and a_3 are indistinguishable, but by using the TBRC a winner is randomly chosen between the two.

3. The Eigenvector Method For Pairwise Voting

When people vote in favor of a candidate A versus a candidate B, they basically rank them and state that for example A is preferred to B, i.e., $A \succ B$. If a population of N individuals vote to select between A and B, if n_1 vote for A and n_2 vote for B, then out of the total number of votes $n_1 + n_2$, $(n_1/n_1 + n_2) \times 100\%$ have voted for A and the remainder for B. Let us assume without loss of generality that $n_1 \geq n_2$. So, for the population of N individuals how much strongly is candidate A preferred over candidate B? The answer is n_1/n_2 if $n_2 > 0$. The matrix of pairwise comparisons representing the relative strength of preference for the candidates is given by

$$\begin{array}{cc} \text{Voting matrix} & \text{Weights} \\ \begin{pmatrix} 1 & \dfrac{n_1}{n_2} \\ \dfrac{n_2}{n_1} & 1 \end{pmatrix} & \begin{pmatrix} \dfrac{n_1}{n_1+n_2} \\ \dfrac{n_2}{n_1+n_2} \end{pmatrix} \end{array}.$$

The voting matrix is a positive reciprocal matrix, i.e., if the entries of the matrix are denoted by a_{ij}, then $a_{ji} = a_{ij}^{-1}$ for all i and j. Note that $a_{ij} \equiv n_i/n_j$.

If instead of two candidates there are m candidates, then because people only vote for one candidate, the voting matrix $A = \{a_{ij}\}$ would be given by

$$\begin{pmatrix} 1 & n_1/n_2 & \cdots & n_1/n_m \\ n_2/n_1 & 1 & \cdots & n_2/n_m \\ \vdots & \vdots & \ddots & \vdots \\ n_m/\,n_1 & n_m/n_2 & \cdots & 1 \end{pmatrix}.$$

Multiplying this matrix by the column vector $\bar{n} = (n_1, n_2, \ldots, n_m)$we get $A\bar{n} = m\bar{n}$. Since the trace of the matrix is equal to m, the trace of a matrix is equal to the sum of all its eigenvalues, and the rank of the A is equal to 1, we conclude that m must be an eigenvalue of A, and it is its largest eigenvalue in modulus. By Perron–Froebenius' theorem[11] it is known that the largest (principle) eigenvalue of a nonnegative real matrix always exists and it is positive and real. In our case, that eigenvalue is equal to m. The right eigenvector associated with the principle eigenvalue is unique to within a multiplicative constant and it is given by

$$\left(\frac{n_1}{N}, \frac{n_2}{N}, \ldots, \frac{n_m}{N}\right)^T,$$

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where N is the total number of voters, i.e.,

$$N = \sum_{i=1}^{m} n_i.$$

Note that the principle right eigenvector yields the correct percentage of votes for each of the candidates.

This model assumes that people vote for one candidate only. However, when a person can vote for more than one candidate, the matrix of paired comparisons may be inconsistent,[14] i.e., the entries do not satisfy the condition $a_{ij}a_{jk} = a_{ik}$ for all i, j and k. In this paper, we are concerned not just with people voting for multiple candidates but ranking them in terms of their preference.

Recall that we denoted by $v_{ij}(\phi)$, the number of voters in a profile ϕ that prefer candidate i to candidate j. Let

$$a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)}, \quad v_{ji}(\phi) > 0.$$

Note that even if the profile ϕ contains all m! preference orders, the voting matrix $A(\phi)$ whose entries are given by

$$a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)}, \quad v_{ji}(\phi) > 0,$$

may not be consistent. So the task is to synthesize, from the ratios of preferences, a vector that best represents the voters' preferences.

We show next that the vector we are seeking is given by the principal right eigenvector of the matrix

$$A(\phi) = \left\{ a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)} \right\}.$$

For the profile B, the matrix of pairwise comparison votes and the corresponding principal eigenvector are given Table 3.

Note that the eigenvector method yields alternative 1 as the winner of the election followed by alternative 3. This result seems consistent with the votes. Although alternatives 2 and 3 appeared to be equally preferred, alternative 1 dominates alternative 2 more than it dominates alternative 3, and hence, alternative 3 receives a larger priority than alternative 2.

Table 3. Pairwise voting matrix.

Voting matrix	PR-eigenvector
$\begin{pmatrix} 1 & 11/3 & 9/5 \\ 3/11 & 1 & 7/7 \\ 5/9 & 7/7 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.526 \\ 0.220 \\ 0.254 \end{pmatrix}$

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3.1. *Properties of the eigenvector method*

In general, when the voting matrix of pairwise comparison ratios is not consistent, it is not possible to infer from

$$a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)} > 1$$

that alternative i defeats alternative j. Let $\bar{w}(\phi)$ be the principle right eigenvector of the matrix $A(\phi)$, i.e., $\mathrm{A}(\phi)\bar{w}(\phi) = \lambda_{\max}\bar{w}(\phi)$.

Definition 2. A reciprocal pairwise voting matrix $A(\phi) = \{a_{ij}(\phi)\}$ satisfies *row dominance* if and only if for any two rows i and j, $a_{ih}(\phi) \geq a_{jh}(\phi)$ or $a_{ih}(\phi) \leq a_{jh}(\phi)$, for all h. Thus, a profile satisfies row dominance if and only if the corresponding reciprocal pairwise voting matrix satisfies row dominance.

Note that because $v_{ih}(\phi) + v_{hi}(\phi) = v_{jh}(\phi) + v_{hj}(\phi) = N$, $a_{ih}(\phi) \geq a_{jh}(\phi)$ implies $v_{ih}(\phi) \geq v_{jh}(\phi)$. Row dominance defines a strong order on the set of alternatives.

Theorem 1. *The eigenvector method on profiles that satisfy row dominance is Condorcet.*

Proof. If a profile satisfies row dominance, then for the corresponding reciprocal pairwise voting matrix one of the rows dominates all others and hence the alternative corresponding to that row is the Condorcet alternative. □

An interesting property that voting methods could satisfy is consistency. Consistency here is not the same as the concept of consistency (or cardinal transitivity) in reciprocal pairwise voting matrices.

Definition 3. A voting method f is said to be *consistent* if and only if given two disjoint profiles $\phi'$and ϕ'' on which the method yields the same consensus ordering, i.e., $f(\phi') = f(\phi'')$, it yields the same consensus ordering on the joint profile $\phi = \phi' \cup \phi''$, i.e., $f(\phi) = f(\phi') = f(\phi'')$.

Lemma 1. *The union of two disjoint profiles that satisfy row dominance and yield the same order on the alternatives satisfies row dominance and yields the same order.*

Proof. Let $A(\phi')$ and $A(\phi'')$ be the reciprocal pairwise voting matrices associated with the profiles. If the profiles ϕ' and ϕ'' satisfy row dominance, then let us assume that for any two alternatives i and j,

$$\frac{v_{ih}(\phi')}{v_{hi}(\phi')} \geq \frac{v_{jh}(\phi')}{v_{hj}(\phi')} \quad \text{and} \quad \frac{v_{ih}(\phi'')}{v_{hi}(\phi'')} \geq \frac{v_{jh}(\phi'')}{v_{hj}(\phi'')} \quad \text{for all } h.$$

$$\frac{v_{ih}(\phi')}{v_{hi}(\phi')} \geq \frac{v_{jh}(\phi')}{v_{hj}(\phi')} \quad \text{and} \quad \frac{v_{ih}(\phi'')}{v_{hi}(\phi'')} \geq \frac{v_{jh}(\phi'')}{v_{hj}(\phi'')}$$

for all h, imply that $v_{ih}(\phi') \geq v_{jh}(\phi')$ and $v_{ih}(\phi'') \geq v_{jh}(\phi'')$ for all h. Thus, we have $v_{ih}(\phi') + v_{ih}(\phi'') \geq v_{jh}(\phi') + v_{jh}(\phi'')$ for all h, which implies that

$$\frac{v_{ih}(\phi') + v_{ih}(\phi'')}{v_{hi}(\phi') + v_{hi}(\phi'')} \geq \frac{v_{jh}(\phi') + v_{jh}(\phi'')}{v_{hj}(\phi') + v_{hj}(\phi'')}$$

for all h, and the joint profile $\phi = \phi' \cup \phi''$ satisfies row dominance. □

Lemma 2. *Given a profile ϕ with corresponding reciprocal pairwise voting matrix*

$$A(\phi) = \left(\frac{v_{ij}(\phi)}{v_{ji}(\phi)}\right)$$

that satisfies row dominance, if for any i and j, $a_{ih}(\phi) \geq a_{jh}(\phi)$, for all h, then $w_i(\phi) \geq w_j(\phi)$, where $\bar{w}(\phi) = (w_i(\phi))^T$ is the principle right eigenvector of the reciprocal pairwise voting matrix $A(\phi)$.

Proof. Follows from the definition of row dominance. Thus, we have

$$w_i(\phi) = \frac{1}{\lambda_{\max}(\phi)} \sum_h \frac{v_{ih}}{v_{hi}} w_h(\phi) \geq \frac{1}{\lambda_{\max}(\phi)} \sum_h \frac{v_{jh}}{v_{hj}} w_h(\phi) = w_j(\phi)$$

□

Theorem 2. *The eigenvector method on profiles that satisfy row dominance is consistent.*

Proof. Let ϕ' and ϕ'' be two disjoint profiles that yield the same consensus ordering. Let $A(\phi')$ and $A(\phi'')$ be the reciprocal pairwise voting matrices associated with the profiles, and let $\bar{w}(\phi')$ and $\bar{w}(\phi'')$ be their principle right eigenvectors. If the eigenvector method yields the same consensus ordering in both profiles, then for any two alternatives i and j $w_i(\phi') \geq w_j(\phi') \Longleftrightarrow w_i(\phi'') \geq w_j(\phi'')$ obtains. In addition, since the profiles satisfy row dominance we have

$$\frac{v_{ih}(\phi')}{v_{hi}(\phi')} \geq \frac{v_{jh}(\phi')}{v_{hj}(\phi')} \quad \text{and} \quad \frac{v_{ih}(\phi'')}{v_{hi}(\phi'')} \geq \frac{v_{jh}(\phi'')}{v_{hj}(\phi'')}$$

for all h if alternative i is preferred to alternative j. By Lemma 1, the joined profile satisfies row dominance and generates the same consensus ordering, and by Lemma 2, $w_i(\phi' \cup \phi'') \geq w_j(\phi' \cup \phi'')$. Thus, the consensus ordering obtained from the joint profile by the eigenvector method is the same as that obtained from the individual profiles, and hence the eigenvector method is a consistent pairwise voting method. □

Definition 4. A voting method satisfies *independence from irrelevant alternatives* (IIA) if the addition/deletion of an alternative to a profile does not alter the consensus ordering obtained from the original profile.

Let ϕ be a profile with m alternatives, let

$$A(\phi) = \left(\frac{v_{ij}(\phi)}{v_{ji}(\phi)}\right)$$

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be the associated reciprocal pairwise voting matrix, and let $\bar{w}(\phi)$ be the corresponding principle right eigenvector of $A(\phi)$. Let ϕ' be the profile resulting from the addition of a new alternative, let

$$A(\phi') = \left(\frac{v_{ij}(\phi')}{v_{ji}(\phi')}\right)$$

be the associated reciprocal pairwise voting matrix, and let $\bar{w}(\phi')$ be the corresponding principle right eigenvector. The matrix $A(\phi')$ is given by

$$A(\phi') = \begin{pmatrix} A(\phi) & \bar{a}_{m+1}(\phi) \\ \bar{a}^{-1}_{m+1}(\phi) & 1 \end{pmatrix}$$

where $\bar{a}_{m+1}(\phi')$ is an $m \times 1$ vector given by

$$\bar{a}_{m+1}(\phi') = \left(\frac{v_{1,m+1}(\phi')}{v_{m+1,1}(\phi')}, \ldots, \frac{v_{m,m+1}(\phi')}{v_{m+1,m}(\phi')}\right)^T.$$

Theorem 3. *The eigenvector method on profiles with row dominance satisfies independence from irrelevant alternatives.*

Proof. Under one common formulation, the IIA axiom states that:

If A is preferred to B out of a choice set, then introducing a third alternative X, thus expanding the choice set, must not make B preferable to A.

Let ϕ be a voting profile with associated reciprocal pairwise voting matrix

$$A(\phi) = \left(\frac{v_{ij}(\phi)}{v_{ji}(\phi)}\right)$$

and principle right eigenvector $\bar{w}(\phi) = (w_i(\phi))^T$. If $A(\phi)$ satisfies row dominance, from Lemma 2, if for any i and j, $a_{ih}(\phi) \geq a_{jh}(\phi)$, for all h, then $w_i(\phi) \geq w_j(\phi)$. Let ϕ' be profile obtained by joining the irrelevant alternative to the profile ϕ. Since an irrelevant alternative is an alternative that it is either dominated or dominates the available alternatives[12] then ϕ' satisfies row dominance and the result follows. □

Corollary 1. *The eigenvector method on profiles that satisfy row dominance satisfies the independence of clones criterion.*

Proof. A clone is an alternative identical to another alternative in the choice set, thus it neither dominates nor it is dominated by all the alternatives. It does not change the preferences of the alternatives and hence the resulting reciprocal pairwise voting matrix still satisfies row dominance, and the result follows.

A condition that is weaker than row dominance is K-dominance. □

Definition 5. A reciprocal pairwise voting matrix $A(\phi)$is row K-dominant if there exists a positive integer K, such that for all $k \geq K$, and any two rows i and j, the kth power of $A(\phi)$, $A(\phi)^k = \{a^{(k)}_{ij}(\phi)\}$, satisfies $a^{(k)}_{ih}(\phi) \geq a^{(k)}_{jh}(\phi)$ or $a^{(k)}_{ih}(\phi) \leq a^{(k)}_{jh}(\phi)$ for all h.

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Lemma 3. *A reciprocal pairwise voting matrix without preference loops is row K-dominant.*

Proof. A preference loop is given by a violation of transitivity, e.g., $a_{ij} > 1, a_{jk} > 1$ but $a_{ik} < 1$. Then $a_{ij} > 1$ would not imply that alternative i dominates alternative j. We proof this lemma by way of a counterexample. Consider the 3×3 matrix given by

$$\begin{pmatrix} 1 & a & 1/a \\ 1/a & 1 & a \\ a & 1/a & 1 \end{pmatrix}$$

with $a>0$. As the matrix is raised to powers, row dominance is not achieved until the limit is attained.[13] Thus, for a matrix to be row K-dominant, it must not have any preference loops. □

Lemma 4. *Let $A(\phi)$be a reciprocal pairwise voting matrix, and let $\bar{w}(\phi) = (w_1(\phi), \ldots, w_m(\phi))^T$ be its principle right eigenvector. Let $A(\phi)^k = \{a_{ij}^{(k)}(\phi)\}$ be the kth power of $A(\phi)$. We have $\lim_{k\to\infty} A(\phi)^k * \mathrm{Diag}[A(\phi)^k]^{-1} = \bar{w}(\phi) * [\bar{w}(\phi)^T]^{-1}$ where*

$$\mathrm{Diag}[A(\phi)^k] = \begin{pmatrix} a_{11}^{(k)}(\phi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{mm}^{(k)}(\phi) \end{pmatrix}.$$

Proof. In [14] it was shown that the limit, as k tends to infinity, of every column of $A(\phi)^k = \{a_{ij}^{(k)}(\phi)\}$ normalized to unity is the same and equal to the principle right eigenvector of $A(\phi)$. Thus, for any column of $A(\phi)^k = \{a_{ij}^{(k)}(\phi)\}$, we have

$$\frac{a_{hi}^{(k)}(\phi)}{\sum_{h=1}^{m} a_{hi}^{(k)}(\phi)} \underset{k\to\infty}{\longrightarrow} w_i(\phi).$$

Thus, we have

$$\lim_{k\to\infty} A(\phi)^k \begin{pmatrix} 1\Big/ a_{11}^{(k)}(\phi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1\Big/ a_{mm}^{(k)}(\phi) \end{pmatrix}$$

$$= \lim_{k\to\infty} A(\phi)^k \begin{pmatrix} \dfrac{\sum_{i=1}^{m} a_{i1}^{(k)}(\phi)}{\sum_{i=1}^{m} a_{i1}^{(k)}(\phi)} \dfrac{1}{a_{11}^{(k)}(\phi)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\sum_{i=1}^{m} a_{im}^{(k)}(\phi)}{\sum_{i=1}^{m} a_{im}^{(k)}(\phi)} \dfrac{1}{a_{mm}^{(k)}(\phi)} \end{pmatrix}$$

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$$
\begin{aligned}
&= \lim_{k\to\infty} A(\phi)^k \begin{pmatrix} \dfrac{1}{\sum_{i=1}^m a_{i1}^{(k)}(\phi)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{1}{\sum_{i=1}^m a_{im}^{(k)}(\phi)} \end{pmatrix} \\
&\quad \lim_{k\to\infty} \begin{pmatrix} \dfrac{\sum_{i=1}^m a_{i1}^{(k)}(\phi)}{a_{11}^{(k)}(\phi)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\sum_{i=1}^m a_{im}^{(k)}(\phi)}{a_{mm}^{(k)}(\phi)} \end{pmatrix} \\
&= \begin{pmatrix} w_1 & w_1 & \cdots & w_1 \\ w_2 & w_2 & \cdots & w_2 \\ \vdots & \vdots & \ddots & \vdots \\ w_m & w_m & \cdots & w_m \end{pmatrix} \begin{pmatrix} 1/w_1 & 0 & \cdots & 0 \\ 0 & 1/w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/w_m \end{pmatrix} \\
&= \begin{pmatrix} 1 & w_1/w_2 & \cdots & w_1/w_m \\ w_2/w_1 & 1 & \cdots & w_2/w_m \\ \vdots & \vdots & \ddots & \vdots \\ w_m/w_1 & w_m/w_2 & \cdots & 1 \end{pmatrix}
\end{aligned}
$$

□

Theorem 4. *Row dominance is a necessary and sufficient condition for the EMPV to satisfy Independence of Irrelevant Alternatives.*

Sufficiency: If the pairwise voting reciprocal matrix is row dominant, then since an irrelevant alternative cannot dominate any of the existing alternatives, it follows that the resulting matrix must also be row dominant, and hence IIE obtains.
Necessity: If the pairwise voting reciprocal matrix is not row dominant, then there exist a pair of alternatives i and j such that $a_{ih}(\phi) \geq a_{jh}(\phi)$ does not hold for all h. Hence, it cannot be guaranteed that $a_{ij}(\phi) > 1 \Rightarrow w_i(\phi) \geq w_j(\phi)$ and IIE does not obtain.

In many voting situations, the withdrawal of a candidate may produce rank reversal preferences unless the profile produced by the voters satisfies row dominance. For example, in Ref. 15 the author gave the following example of rank reversal using the AHP. However, our method does not lead to such a result because the profile generated is row dominant. Perez considered the following profile

	(1000)	(2000)
a_1	500	520
a_2	260	745
a_3	240	735

Table 4. The pairwise voting matrix.

	a_1	a_2	a_3
a_1	1	$\frac{1020}{1005}$	$\frac{1020}{975}$
a_2	$\frac{1005}{1020}$	1	$\frac{1005}{975}$
a_3	$\frac{975}{1020}$	$\frac{975}{1005}$	1

Table 5. Profile and pairwise voting matrix.

Rank	(40)	(35)	(25)
1	a_1	a_2	a_3
2	a_2	a_3	a_1
3	a_3	a_1	a_2

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 1 & 65/35 & 40/60 \\ 35/65 & 1 & 75/25 \\ 60/40 & 25/75 & 1 \end{pmatrix} \end{array}.$$

in which 1000 (2000) voters selected among three candidates and 500 (520) selected a_1, 260 (745) selected a_2 and 240 (735) selected a_3. Here each voter only selects one candidate. The reciprocal pairwise voting matrix generated by this profile is given in Table 4. that yields the principle right eigenvector (0.34, 0.335, 0.325). Note that if one of the candidates withdraws, the result remains unaltered because the profile yielded a row dominant reciprocal pairwise voting matrix.

Consider now the following profile and its corresponding pairwise voting matrix in Table 5. This matrix is not row dominant and consequently independence from irrelevant alternatives is not satisfied.

4. Pairwise Voting with Intensity of Preferences

In elections, voting is a modified form of ranking and all the votes are equally important. However, there are situations similar to voting in which the votes are not just ordinal but each voter expresses an intensity of preference for the different candidates, e.g., ranking projects for funding. The voters, in this case they are called referees, score the different projects according to some criteria. The scores are then used to rank the projects.

4.1. *Ranking with complete pairwise comparisons*

Ranking can be considered a limiting condition of pairwise comparisons. Consider two projects a_1 and a_2. Comparing them according to a criterion, we can express how

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strongly we prefer one project to the other. For example, if a_1 is preferred to a_2 with intensity a the result is a reciprocal matrix of pairwise comparisons given by

$$\begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} 1 & a \\ a^{-1} & 1 \end{pmatrix} \end{array}.$$

This matrix has a principle right eigenvector given by

$$\begin{pmatrix} \dfrac{a}{1+a} \\ \dfrac{1}{1+a} \end{pmatrix}$$

that converges to the vector $(1,0)^T$ as $a \to \infty$. Thus, ordinal ranking of the two projects is equivalent to pairwise comparing them with an intensity of infinity of one project over another.

Given a profile

$$\phi = \begin{pmatrix} (n_{1>2}) & (n_{2>1}) \\ a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$$

the pairwise voting matrix is given by

$$\begin{array}{c} a_1 \\ a_2 \end{array} \begin{pmatrix} 1 & n_{1>2}/n_{2>1} \\ n_{2>1}/n_{1>2} & 1 \end{pmatrix}.$$

If all the voters that prefer a_1 to a_2 have an intensity of preference a_{12}, then we can represent the pairwise voting matrix as follows, so that when $a_{12} \to \infty$ we obtain the pairwise voting matrix given above:

$$\begin{pmatrix} 1 & \dfrac{\left(\frac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi)}{\left(\frac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)} \\ \dfrac{\left(\frac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)}{\left(\frac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi)} & 1 \end{pmatrix}.$$

The principle right eigenvector of this matrix is given by

$$\begin{pmatrix} \dfrac{\left(\frac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi)}{\left(\frac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi) + \left(\frac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)} \\ \dfrac{\left(\frac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)}{\left(\frac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi) + \left(\frac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)} \end{pmatrix}.$$

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If all the voters have different intensity of preference $a_{i>j}^{(k)}$, then the pairwise voting matrix is given by

$$\begin{pmatrix} 1 & \dfrac{\displaystyle\sum_{k=1}^{n_{1>2}(\phi)}\left(\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}\right)}{\displaystyle\sum_{k=1}^{n_{2>1}(\phi)}\left(\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}\right)} \\ \dfrac{\displaystyle\sum_{k=1}^{n_{2>1}(\phi)}\left(\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}\right)}{\displaystyle\sum_{k=1}^{n_{1>2}(\phi)}\left(\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}\right)} & 1 \end{pmatrix}.$$

Finally, if we compare m projects, the pairwise voting matrix with intensity of preferences would be given by

$$\begin{pmatrix} 1 & \dfrac{\displaystyle\sum_{k=1}^{n_{1>2}(\phi)}\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}}{\displaystyle\sum_{k=1}^{n_{1<2}(\phi)}\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}} & \cdots & \dfrac{\displaystyle\sum_{k=1}^{n_{1>m}(\phi)}\dfrac{a_{1m}^{(k)}}{a_{1m}^{(k)}+1}}{\displaystyle\sum_{k=1}^{n_{1<m}(\phi)}\dfrac{a_{m1}^{(k)}}{a_{m1}^{(k)}+1}} \\ - & 1 & \cdots & \dfrac{\displaystyle\sum_{k=1}^{n_{2>m}(\phi)}\dfrac{a_{2m}^{(k)}}{a_{2m}^{(k)}+1}}{\displaystyle\sum_{k=1}^{n_{2<m}(\phi)}\dfrac{a_{m2}^{(k)}}{a_{m2}^{(k)}+1}} \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \cdots & 1 \end{pmatrix},$$

where $n_{i>j}(\phi)$ and $a_{ij}^{(k)}$ represent the number of voters that prefer i to j and the intensity with which the kth voter prefers i to j, respectively. Note that if $a_{ij}^{(k)} \to \infty$, for all i and j, the pairwise voting matrix converges to the matrix

$$\begin{pmatrix} 1 & w_{12}(\phi) & \cdots & w_{1m}(\phi) \\ w_{21}(\phi) & 1 & \cdots & w_{2m}(\phi) \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1}(\phi) & w_{m2}(\phi) & \cdots & 1 \end{pmatrix},$$

where

$$w_{ij}(\phi) = \frac{n_{i>j}(\phi)}{n_{j>i}(\phi)}, \quad n_{j>i}(\phi) > 0.$$

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If that matrix is row dominant, then the principle right eigenvector is Condorcet.

Example. Consider the profile at the beginning of the paper.

$$B = \begin{array}{c} \begin{array}{cccc} (4) & (3) & (5) & (2) \end{array} \\ \begin{pmatrix} a_1 & a_2 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & a_1 & a_2 & a_1 \end{pmatrix} \end{array}.$$

Let us assume that the intensity of the preferences of the individual voters is given in Table 6.

Note that the individual priorities yield the order of preferences of the individual voters. If we were to combine the voting preferences as shown above, we would obtain the pairwise preference matrices and corresponding priorities given in Table 7.

Note that the resulting priorities are close to the ones obtained without intensity of preferences.

Table 6. Intensity of voting preferences for profile B.

Orderings			Voter	pairwise preference matrices			Priorities
1	2	3	1	1	3	5	0.617504
				0.333333	1	5	0.296865
				0.2	0.2	1	0.085631
			2	1	5	7	0.71471
				0.2	1	5	0.218494
				0.142857	0.2	1	0.066796
			3	1	2	5	0.559065
				0.5	1	5	0.352189
				0.2	0.2	1	0.088746
			4	1	4	8	0.698615
				0.25	1	5	0.237042
				0.125	0.2	1	0.064343
2	3	1	5	1	0.2	0.333333	0.104729
				5	1	3	0.636986
				3	0.333333	1	0.258285
			6	1	0.2	0.333333	0.097169
				5	1	5	0.700711
				3	0.2	1	0.20212
			7	1	0.142857	0.111111	0.058277
				7	1	2	0.558884
				9	0.5	1	0.382839
1	3	2	8	1	7	3	0.649118
				0.142857	1	0.2	0.071927
				0.333333	5	1	0.278955
			9	1	5	3	0.63093
				0.2	1	0.2	0.087493
				0.333333	4	1	0.281577
			10	1	6	3	0.664729
				0.166667	1	0.2	0.081629

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Table 6. (*Continued*)

Orderings				Voter pairwise preference matrices			Priorities
				0.333333	3	1	0.253642
			11	1	4	3	0.625898
				0.25	1	0.2	0.100716
				0.333333	3	1	0.273386
			12	1	5	4	0.705023
				0.2	1	0.2	0.088827
				0.25	2	1	0.20615
3	2	1	13	1	0.2	0.111111	0.058111
				5	1	0.2	0.206695
				9	5	1	0.735193
			14	1	0.333333	0.2	0.09842
				3	1	0.333333	0.242724
				5	4	1	0.658856

Table 7. Priorities with and without intensities.

Synthesized pairwise matrix				Priorities	
				w/Intensities	w/o Intensities
1	1	3.015925	1.749328	0.534	0.526
2	0.331573	1	1.027607	0.214	0.22
3	0.571648	0.973134	1	0.252	0.254

4.2. *Ranking with incomplete pairwise comparisons*

Let us now assume that the voters do not produce all possible orderings of the candidates to make it possible to have

$$w_{ij}(\phi) = \frac{n_{i>j}(\phi)}{n_{j>i}(\phi)}, \quad n_{j>i}(\phi) > 0$$

for all i and j. In other words, there exist some i and j for which $n_{i>j}(\phi) > 0$ but $n_{j>i}(\phi) = 0$. Thus, in the matrix

$$\begin{pmatrix} 1 & w_{12}(\phi) & \cdots & w_{1i}(\phi) & \cdots & w_{1j}(\phi) & \cdots & w_{1m}(\phi) \\ w_{21}(\phi) & 1 & \cdots & w_{2i}(\phi) & \cdots & w_{2j}(\phi) & \cdots & w_{2m}(\phi) \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ w_{i1}(\phi) & w_{i2}(\phi) & \cdots & 1 & \cdots & w_{ij}(\phi) & \cdots & w_{im}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ w_{j1}(\phi) & w_{j2}(\phi) & \cdots & w_{ji}(\phi) & \cdots & 1 & \cdots & w_{jm}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ w_{m1}(\phi) & w_{m2}(\phi) & \cdots & w_{mi}(\phi) & \cdots & w_{mj}(\phi) & \cdots & 1 \end{pmatrix}$$

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the (i, j) entry would be as given below

$$\begin{pmatrix} 1 & w_{12}(\phi) & \cdots & w_{1i}(\phi) & \cdots & w_{1j}(\phi) & \cdots & w_{1m}(\phi) \\ w_{21}(\phi) & 1 & \cdots & w_{2i}(\phi) & \cdots & w_{2j}(\phi) & \cdots & w_{2m}(\phi) \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ w_{i1}(\phi) & w_{i2}(\phi) & \cdots & 1 & \cdots & \infty & \cdots & w_{im}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ w_{j1}(\phi) & w_{j2}(\phi) & \cdots & 0 & \cdots & 1 & \cdots & w_{jm}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ w_{m1}(\phi) & w_{m2}(\phi) & \cdots & w_{mi}(\phi) & \cdots & w_{mj}(\phi) & \cdots & 1 \end{pmatrix}.$$

To avoid this situation, we could introduce a fictitious voter in the profile which prefers j to i and has no other preferences for the other candidates. The pairwise comparison voting matrix would now be given by

$$\begin{pmatrix} 1 & w_{12}(\phi) & \cdots & w_{1i}(\phi) & \cdots & w_{1j}(\phi) & \cdots & w_{1m}(\phi) \\ w_{21}(\phi) & 1 & \cdots & w_{2i}(\phi) & \cdots & w_{2j}(\phi) & \cdots & w_{2m}(\phi) \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ w_{i1}(\phi) & w_{i2}(\phi) & \cdots & 1 & \cdots & n_{ij}(\phi)/1 & \cdots & w_{im}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ w_{j1}(\phi) & w_{j2}(\phi) & \cdots & 1/n_{ij}(\phi) & \cdots & 1 & \cdots & w_{jm}(\phi) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ w_{m1}(\phi) & w_{m2}(\phi) & \cdots & w_{mi}(\phi) & \cdots & w_{mj}(\phi) & \cdots & 1 \end{pmatrix}.$$

For a two-candidate profile, if only one of the candidates receives all the votes, then the priority of the candidate with all the votes is equal to one. For a three-candidate profile, if the profile does not have all possible orderings and some of the candidates do not get any votes such as in the following profile

$$\phi = \begin{pmatrix} (n_{1>2>3}) & (n_{3>1>2}) \\ a_1 & a_3 \\ a_2 & a_1 \\ a_3 & a_2 \end{pmatrix}$$

the pairwise matrix of voting preferences is given by

$$A(\phi) = \begin{pmatrix} 1 & (n_{1>2>3} + n_{3>1>2})/\,0 & n_{1>2>3}/n_{3>1>2} \\ & 1 & n_{1>2>3}/n_{3>1>2} \\ & & 1 \end{pmatrix}.$$

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To find the priorities of the candidates, we could add a fictitious voter that ranks a_2 ahead of a_1. The profile would now be as given below

$$\phi = \begin{pmatrix} (n_{1>2>3}) & (n_{3>1>2}) & (1) \\ a_1 & a_3 & a_2 \\ a_2 & a_1 & a_1 \\ a_3 & a_2 & \end{pmatrix}$$

and the pairwise voting matrix would be given by

$$A(\phi) = \begin{pmatrix} 1 & (n_{1>2>3} + n_{3>1>2})/1 & n_{1>2>3}/n_{3>1>2} \\ & 1 & n_{1>2>3}/n_{3>1>2} \\ & & 1 \end{pmatrix}$$

For $n_{1>2>3} + n_{3>1>2}$ that is sufficiently large, the addition of an extra voter will not make a difference on the priorities of the candidates. So the problem would be if the number of voters is small.

5. Conclusions

Voting with intensity of preferences softens attitudes, but for large populations produces similar results as voting without intensity of preferences. Thus, allowing voters to rank candidates, and using the eigenvector method of pairwise comparisons to obtain the winner of the contest is almost equivalent to letting them compare voters with intensity of preferences. Intensity of preferences plays a fundamental role when voters compare candidates in terms of different criteria, such as the economy, terrorism, immigration and so on. Then, ranking candidates for each dimension is appropriate to extract priorities, but these priorities cannot be combined unless we compare the importance of the dimensions. One could obtain priorities for the dimensions by having the group of voters rank them, i.e., building a profile. However, there is a problem with this approach if there is no diversity of thinking, e.g., all the voters give the same ranking for the dimensions. Then priorities cannot be derived and the priorities for the candidates under each dimension cannot be synthesized to produce a winner. This lets us conclude that intensity of preferences not only cannot be avoided but does not violate democracy, and hence, they should be part of the voting process.

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